Problem Set 5
Exercises on Diagonalization
CSCI 6114 Fall 2023

Joshua A. Grochow

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Read before Thursday

- Arora & Barak Sections 3.1–3.3

Exercises

1. Show that the set of real numbers whose decimal expansions contain infinitely many 1s is not countable.

2. Show that there is no fixed k such that $\mathsf{NP} \subseteq \mathsf{DTIME}(O(n^k))$. *Hint:* Use the Time Hierarchy Theorem. (Compare with Kannan's Theorem from last week.)

3. Prove a Space Hierarchy Theorem: if $S(n)$ is fully space-constructible, and $S_2(n) < o(S(n))$, then $\mathsf{DSPACE}(S_2(n)) \subsetneq \mathsf{DSPACE}(S(n))$. (If you have difficulty proving such a tight version, first try to prove something weaker, e.g. assuming $S_2(n) < o(S(n)^{1/2})$.)

4. (At home) Given a deterministic, multi-tape Turing machine M, the number of *tape reversals* on input x is the total number of times the tape heads change direction (the last time it moved, it moved left, and now it moves right, or vice versa). As a complexity measure, tape reversals are closely related to parallel time complexity (Pippinger, FOCS '79). Prove as tight a Tape Reversal Hierarchy Theorem as you can.

5. (a) A relativizable complexity class $\mathcal{C}^{\square}$ is called *self-low* if $\mathcal{C}^{\mathcal{C}} = \mathcal{C}$. (e.g. P is self-low, but NP is not self-low unless PH collapses - do you see why?) Show that PSPACE is self-low.

   (b) Use part (a) to show that $\mathsf{P}^{\mathsf{PSPACE}} = \mathsf{NP}^{\mathsf{PSPACE}}$. (Thus: if $\mathsf{P} \neq \mathsf{NP}$, this fact would not relativize [to all oracles], so it requires non-relativizing techniques to prove.) *Hint:* Recall the relationship between NP and PSPACE, and see that this relationship relativizes.

Resources

- Arora & Barak Theorem 3.3 for the Nondeterministic Time Hierarchy Theorem.
- Homer & Selman Theorem 5.15 is the Space Hierarchy Theorem
- Kozen, Indexing of subrecursive classes *Theoret. Comput. Sci.* 1980, shows that for certain complexity classes, any separation can be proved by a kind of diagonalization argument.
- Chapter 1 of Gems of TCS is a more advanced form of diagonalization, which is used to prove that there are uncomputable problems that are strictly weaker than the Halting problem (i.e. problems in between COMP and CE-complete).